

# Thermostatistics with minimal length uncertainty relation

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## Abstract

Existence of minimal length is suggested in any quantum theory of gravity such as string theory, double special relativity and black hole physics. One way to impose minimal length is deforming Heisenberg algebra in phase space which is called Generalized Uncertainty Principle (GUP). In this paper, we develop statistical mechanics in GUP framework. Our method is quite general and does not need to fix the generalized coordinates and momenta. We define general transformation in phase space which transforms usual Heisenberg algebra to a deformed one. In this method, quantum gravity effects only acts on the structure of phase space and we relate these effects to the density of states. We find an interesting phenomenon in Maxwell-Boltzmann statistics which has not a classical analogy. We show that there is an upper bound for the number of excited particles in the limit of high temperature which implies to the condensation. Also we study modification of Bose-Einstein condensation and the completely degenerate gas.

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## 1 Introduction

In Hamiltonian viewpoint of classical mechanics, canonical equations of motion can be represented by the Poisson brackets in which the coordinates  $x_i$  and their conjugate momenta  $p_j$  obey the Poisson algebra  $\{x_i, p_j\} = \delta_{ij}$ . Physically, these relations may be interpreted as the measurement of the position and momentum of a moving particle simultaneously. This means that the coordinates of each point of the corresponding phase space can be determined without any uncertainty. The transition to the quantum mechanics is straightforward. The classical dynamical variables should be replaced by their hermitian operators counterparts in Hilbert space and the Poisson brackets with the Dirac commutators. Hence, the above Poisson algebra takes the form of the Heisenberg algebra  $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$ . A consequence of such a algebra is that it divides the phase space to pixels proportional to the Planck constant and thus, the position and the momentum of particles can not be determined simultaneously. This is what we know as the uncertainty principle in the usual quantum mechanics. On the other hand, one of the most important predictions of the theories which deal with quantum gravity is that there exists a minimal length below which no other length can be observed [1]. One of the interesting features of the existence of such a minimum length is the modification it makes to the standard commutation relation between position and momentum in ordinary quantum mechanics which is called the generalized uncertainty principle (GUP) [2]. In one dimension the simplest form

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of such relations can be written as

$$\Delta X \Delta P \geq \frac{\hbar}{2} (1 + \beta(\Delta P)^2 + \gamma), \quad (1)$$

where  $\beta$  and  $\gamma$  are positive and independent of  $\Delta X$  and  $\Delta P$ , but may in general depend on the expectation values  $\langle X \rangle$  and  $\langle P \rangle$ . The usual Heisenberg uncertainty relation can be recovered in the limit  $\beta = \gamma = 0$ . It is easy to see that this equation implies a minimum position uncertainty of  $(\Delta X)_{min} = \hbar\sqrt{\beta}$ . For a more general discussion on such deformed Heisenberg algebras, especially in three dimensions, see [3]. Now, it is possible to realize equation (1) from the following commutation relation between position and momentum operators

$$[\hat{X}_i, \hat{P}_j] = i\hbar(1 + \beta\mathbf{P}^2) \delta_{ij}, \quad (2)$$

where  $\mathbf{P}^2 = P_i P^i$  and we take  $\gamma = \beta < \mathbf{P} >^2$ . Also, assuming that

$$[\hat{P}_i, \hat{P}_j] = 0, \quad (3)$$

the commutation relations for the coordinates are obtained as

$$[\hat{X}_i, \hat{X}_j] = 2i\hbar\beta (\hat{P}_i \hat{X}_j - \hat{P}_j \hat{X}_i), \quad (4)$$

which means that in more than one dimension, GUP naturally implies a noncommutative geometric generalization of position space.

In a statistical mechanics point of view, the microstate of a given classical system may be defined by  $3N$  position coordinates  $X_1, \dots, X_{3N}$  and  $3N$  momenta coordinates  $P_1, \dots, P_{3N}$ , where  $N$  is the number of particles in the system. In a geometric picture, the set of coordinates  $(X_i, P_j)$ , where  $i = 1, \dots, 3N$ , may be considered as a point in a  $6N$  dimensional space, the so-called phase space of the system. Since the coordinates  $X_i$  and  $P_i$  vary with time, the dynamics of the whole system can be determined with the help of the Hamiltonian equations of motion for each of these coordinates, that is,  $\dot{X}_i = \{X_i, H\}$  and  $\dot{P}_i = \{P_i, H\}$ , where  $H(X_i, P_j)$  is the Hamiltonian of the system. Therefore, the dynamical behavior of the system may be viewed as a continuous trajectory of the phase point  $(X_i, P_i)$  in the phase space. On the other hand, it is well known that the correct description of such a system and its evolution requires quantum mechanics. In this case, the quantum particle has no well-defined trajectory in the phase space. This is because that at a given time  $t$ , there is no longer an exact position, but a typical extent of the wave function inside which this particle can be found. Another way of understanding this issue is to state the Heisenberg uncertainty principle which is a result of the commutation relations between position and momentum operators. In this sense, if the position coordinate  $X_i$  is known up to  $\Delta X_i$ , then its conjugate momentum  $P_i$  cannot be determined with an accuracy better than  $\Delta P_i$ , such that  $\Delta X_i \Delta P_i \geq \hbar/2$ . The phase space is therefore divided into cells of area of the order of  $h^{3N}$ . The accuracy on the phase space trajectory is thus limited and this produces a discretization of the phase space. Now, what happened if one takes into account the GUP considerations instead of the ordinary uncertainty principle? The motivation of this question is that the GUP scheme relies on a modification of the canonical prescriptions and, in this respect, can be reliably applied to any dynamical system. Over the past few years, a number of works have been done, in the area of statistical mechanics in the GUP framework. For instance, the thermodynamics of the ideal gas and ultra-relativistic gas in micro-canonical ensemble in the GUP framework are studied in [4]. For harmonic oscillators and ideal gases in canonical ensembles see [5]. The deformed density matrix is studied in [6] and modified uncertainty relation for inverse temperature and internal energy is addressed in [7]. Black body radiation with minimal length effects is considered in [8]. For thermodynamics in doubly special relativity theory which also predicts a minimal length, see [9]. Thermodynamics in the framework of non-commutative Heisenberg algebra (q-deformed) is investigated in [10].

In this paper, our aim is to study the aspects related to the application of GUP in the framework of classical and quantum statistical mechanics. Our study covers the thermodynamics of the fermionic and bosonic systems when their underlying canonical structures are of the type of GUP-deformed. We deal with the density of deformed phase space and study the effects of existence of minimal length on the density of states. We show that there exist a general transformation which transforms the usual Heisenberg algebra to a deformed one and after we convince ourselves to the existence of such a transformation, we obtain a density of state with the help of the Jacobian of the transformation. Then, we study the thermodynamics of the ideal gas and extreme relativistic gas with the help of density of deformed phase space. The Bose-Einstein condensation and the completely degenerate gas are the case studies which we will deal with them in the GUP framework. Also, we find an interesting phenomenon in the Maxwell-Boltzmann statistics which we interpret it as the *gravitational condensation* and we show that it occurs for the ideal gas and the extreme relativistic gas in the same manner.

## 2 Statistical mechanics

Usually introducing the GUP version of a quantum theory is done by replacing the ordinary commutation relations by their generalized counterparts (2) and (4). However, in what follows we act differently and define a transformation in the phase space which transforms the usual Heisenberg algebra to a deformed one. In this picture, GUP effects only change the density of the phase space of a statistical system. As we will see the volume of phase space becomes larger while its degrees of freedom decreases.

The microstates of any physical system are determined with quantum mechanics and its energy levels should be obtained from the Schrödinger equation. In GUP framework, the Schrödinger equation becomes a non-linear or higher order differential equation and it is not easy to solve it. For example, for the wave function and energy spectrum of harmonic oscillator see [2] and [11]. Particular non-linear Schrödinger equation in GUP framework is proposed in [12] and solved for quantum bouncer in [13]. For the higher order modified schrödinger equation for quantum mechanical systems, see [14]. Of course, solving the higher order differential equation has some dramatic consequences because of the mathematical difficulty due to non-linearity and initial values for higher order differential equations. In our method it is not necessary to solve the modified Schrödinger equation, instead, we work in semi-classical approximation and we use the classical Hamiltonian in deformed phase space. In quantum picture, energy of a given macrostate is a summation of the energies of all corresponding microstates. On the other hand, energy of the macrostate is given by the Hamiltonian which is a continuous function of the phase space variables. It is customary to approximate the summation over the energies of the microstates by the integral over all phase space variables. As we have mentioned, the phase space is constructed by the fundamental cells as order of Planck constant  $\hbar$ , [15]. We show that the fundamental cell becomes larger and also has a momentum dependence as  $\hbar_{GUP} = \hbar(1 + \beta P^2)$ . Then, we can approximate summation over energies of microstates by integral over all deformed phase space with spotting lattice structure of phase space. Let us now explain our approach to transition from ordinary phase space to deformed phase space and also show the momentum dependence of the  $\hbar_{GUP}$ . In ordinary quantum mechanics, the coordinates  $x_i$  and momenta  $p_i$  satisfy Heisenberg algebra  $[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}$  where its classical counterpart is the Poisson algebra  $\{x_i, p_j\} = \delta_{ij}$ . In the GUP framework, these relations take the form

$$\{X_i, X_j\} = 2\beta(P_i X_j - P_j X_i), \quad (5)$$

$$\{X_i, P_j\} = (1 + \beta P^2) \delta_{ij}, \quad (6)$$

$$\{P_i, P_j\} = 0. \quad (7)$$

According to the Darboux theorem [16], it is always possible to find canonically conjugate variables  $X_i(x, p)$  and  $P_i(x, p)$  such that they satisfy relation (6) and (7). However, we would like to find variables  $X_i$  and  $P_i$  which satisfy also relation (5) simultaneously. In Appendix A, we convince ourselves that it is always possible to find generalized variables  $X_i$  and  $P_i$  which satisfy relations (5), (6) and (7) simultaneously. This possibility allows us to define general transformation

$$(x_i, p_i) \rightarrow (X_i(x, p), P_i(x, p)), \quad (8)$$

which transforms the usual Heisenberg algebra to a deformed one. Now, we only should have the Jacobian of this transformation for our study. The Jacobian  $J$  of the transformation can be expressed in terms of poisson brackets in  $2\mathcal{N}$ -dimensional phase space ([5], see also Appendix A),

$$J = \frac{\partial(X_i, P_i)}{\partial(x_i, p_i)} = \prod_{i=1}^{\mathcal{N}} \{X_i, P_i\} = (1 + \beta P^2)^{\mathcal{N}}. \quad (9)$$

In Appendix A we show that the relation (5) does not contribute in Jacobian, thus we miss the effects of non-commutativity in our study. Having the Jacobian of the transformation at hand, we can define the state density  $a(\varepsilon) d\varepsilon$  in GUP framework by which we may approximate the summation of energy spectrums with an integral over deformed phase space. In the limit of large volume  $V$ , we can replace summation over energy spectrums with an integral as  $\sum \rightarrow \int \frac{d\omega}{\omega_0}$ , where  $d\omega = d^{\mathcal{N}}x d^{\mathcal{N}}p$  and  $\mathcal{N}$  is the number of degrees of freedom. Here,  $\omega_0$  is a fundamental volume and is given by  $\omega_0 = h^{\mathcal{N}}$ , where  $h$  is the Planck constant. We define ordinary density of state as  $\sum \rightarrow \int a_0(\varepsilon) d\varepsilon$ . Now, we can define the density of states in GUP framework with the help of transformation (8). In terms of the generalized variables  $X_i$  and  $P_i$ , the density of states for the deformed phase space becomes

$$a(\varepsilon) d\varepsilon = \frac{1}{h^{\mathcal{N}}} \frac{d^{\mathcal{N}}X d^{\mathcal{N}}P}{J} = \frac{d^{\mathcal{N}}X d^{\mathcal{N}}P}{[h(1 + \beta P^2)]^{\mathcal{N}}}. \quad (10)$$

From the above relation, it is clear that the fundamental volume element of the corresponding phase space takes the form  $\hbar(1 + \beta P^2)$  which is larger in comparison with the ordinary case. Thus, the number of accessible microstates for the system will decrease in GUP framework. Relation (10) plays an important role in our approach and we formulate the ensemble theory in GUP framework with the help of it in this section.

## 2.1 Micro-canonical ensemble

In equilibrium statistical mechanics, a system is supposed to be in the thermal equilibrium and different ensembles have different density of microstates in phase space. In pervious section we obtain density of the deformed phase space, then it is easy to generalize ensemble density for any ensemble in GUP framework. In micro-canonical, the energy of a given macrostate remains constant along the thermodynamical processes. This condition imposes the energy constraint to the system. Under this condition the probability of all microstates is same and density of state is constant. We assume that the density of states to be  $\rho(x, p)^{mce} \propto \delta(H(x, p) - E)$  (see [17]), where  $\delta$  is the Dirac delta function, which clearly satisfies two conditions, the same probability for all microstates and energy constraint. Now we define the number of microstates for  $N$  particles in GUP framework as

$$\Sigma(\beta) = \frac{1}{h^{3N}} \int \int \frac{d^{3N}X d^{3N}P}{(1 + \beta P^2)^{3N}} \prod_{i=1}^N \delta(H_i(X, P) - E'_i), \quad (11)$$

where  $E' = E'(\beta, E)$  is energy of each particle in GUP framework<sup>1</sup>. It is clear that the number of accessible microstates decrease because of the GUP correction in denominator which may be

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<sup>1</sup>We work in unit where  $k_b = 1 = c$ , where  $k_b$  is the Boltzmann constant and  $c$  is the speed of light.

interpreted as a lost of information (see [18] and [4]). Now, the entropy of the system can be easily obtained from its standard definition as

$$\mathcal{S} = \ln \Sigma(\beta). \quad (12)$$

There is an interesting case, because of the decreasing number of microstates, entropy decreases and tends to zero in high energy regime (high temperature limit). In the very early theory of universe zero entropy coincides with preliminary singularity of universe (highest level of energy) and all the information miss there as we expect from the big-bang theory. Thermodynamics of micro-canonical ensemble can be obtained from entropy (12).

## 2.2 Canonical ensemble

In canonical ensemble the energy of a system is variable and system can exchange energy. It means that the energy of the microstates determine the probability of them in phase space. Then, the density of microstates should be a function of energy and we know it is proportional to Boltzmann factor:  $\rho(x, p)^{ce} \propto \exp(-\frac{H(x, p)}{T})$ . The partition function of the system is given by

$$Z = \sum_{\epsilon} \exp(-\frac{\epsilon}{T}), \quad (13)$$

where  $\epsilon$  is the energy of the microstates and is a solution of Schrödinger equation. In fact we should solve modified Schrödinger equation in GUP framework and substitute energy of microstates in (13). However, in our approach we approximate summation over energy levels by an integral with the help of relation (10). Therefore, for a system with  $N$  particles we get

$$Z_N(T, V, \beta) = \frac{1}{h^{3N}} \int \int \frac{d^{3N} X d^{3N} P}{(1 + \beta P^2)^{3N}} \exp(-\frac{H(X, P)}{T}). \quad (14)$$

Now, all of the thermodynamical quantities in GUP framework can be obtain from the partition function [5].

## 2.3 Grand canonical ensemble

In grand canonical ensemble, in addition of its energy, the system can be exchange its particles. Then we expect the probability of microstates depend to the number of particles and energy. As same as the pervious section the grand partition function in GUP framework is

$$\Xi(T, V, N, \beta) = \sum_{N=0}^{\infty} z^N Z_N(T, V, \beta), \quad (15)$$

where  $z = e^{\mu/T}$  is the fugacity of the system and  $\mu$  is the chemical potential. Again all the thermodynamical quantities can be obtained from grand canonical partition function. On the other hand, one may work with *mean occupation number*,  $\langle n_{\epsilon} \rangle$  instead of grand partition function (15) which can be defined as

$$\langle n_{\epsilon} \rangle = \frac{1}{z^{-1} e^{\epsilon/T} + \epsilon}, \quad (16)$$

where  $\epsilon$  is a constant with the values  $\epsilon = 1$  (for Fermi-Dirac statistics) and  $\epsilon = -1$  (for Bose-Einstein statistics). The particle density and pressure can be obtain from  $\langle n_{\epsilon} \rangle$  by definitions

$$N = \sum_{\epsilon} \langle n_{\epsilon} \rangle, \quad (17)$$

$$\frac{\mathcal{P}V}{T} = \frac{1}{\epsilon} \sum_{\epsilon} \ln(1 + \epsilon z e^{-\epsilon/T}). \quad (18)$$

In the limit of Large volume, we can replace the above summation by an integral over all deformed phase space with the help of density of state in GUP framework (10). Fortunately, there is a closed form for the solution in Maxwell-Boltzmann statistics correspond to the limit  $z \rightarrow 0$ . In this limit we have a classical limit of minimal length and we only neglect the quantum effects, but the GUP effects still exist. On the other hand, in fully quantum area ( $z \rightarrow 1$ ), we expect to find the purely quantum phenomenon such as Bose-Einstein condensation. In these limit, there is an analytical solution for any value of  $z$  in term of Fermi-Dirac and Bose-Einstein functions (see Appendix B). We study some consequences of minimal length for two gaseous systems, *Ideal gas* and *Extreme relativistic gas* in next section and consider thermodynamics of them in Maxwell-Boltzmann, Fermi-Dirac and Bose-Einstein statistics.

### 3 Maxwell-Boltzmann statistics

To study the effects of minimal length in thermodynamics, we can exert GUP condition with the help of relation (10), but we should note when we approximate summation in relation (17) by integral, the contribution of the first term which corresponds to the ground state energy of particles ( $\epsilon = 0$ ) missed. Thus, we decompose the first term as

$$N_e = N - N_0 = \int < n_{\epsilon} > a(\epsilon) d\epsilon, \quad (19)$$

where  $N_e$  and  $N_0$  denote the number of particles in the excited states ground state respectively. The value of  $N_0$  which corresponds to  $\frac{z}{1-z}$  is negligible for classical limit  $z \rightarrow 0$  and becomes important in the limit of fully quantum area  $z \rightarrow 1$ . However, we keep this term in both cases. In Maxwell-Boltzmann statistics, the quantum effects are negligible and the type of particles (fermion or boson) is not important. Then, our start point is relations (17) and (18) according to which we have

$$N^{cl} = \lim_{z \rightarrow 0} \left( \sum_{\epsilon} < n_{\epsilon} > \right) = z \sum_{\epsilon} e^{-\epsilon/T}, \quad (20)$$

and

$$\frac{\mathcal{P}^{cl}V}{T} = \lim_{z \rightarrow 0} \left( \frac{1}{\epsilon} \sum_{\epsilon} \ln(1 + \epsilon z e^{-\epsilon/T}) \right) = z \sum_{\epsilon} e^{-\epsilon/T}, \quad (21)$$

which resemble Maxwell-Boltzmann distribution as we expected for classical systems. It is seen that both of the above relations have the same value in this limit. In fact, they are a result of equation of state in classical statistical mechanics and by combining them, we get

$$\mathcal{P}^{cl}V = N^{cl}T, \quad (22)$$

which is a familiar classical equation of state. Thus the form of equation of states preserve for statistical systems in GUP theory. It is important to realize that the pressure  $\mathcal{P}^{cl}$  and  $N^{cl}$  are in GUP framework and have different forms in comparison with their ordinary cases. Now, we calculate  $\mathcal{P}^{cl}$  and  $N^{cl}$  for two gaseous systems, the ideal gas and extreme relativistic gas.

### 3.1 Ideal gas

We consider a gaseous system consisting of  $N$  non-interacting particles confined in volume  $V$  which can sharing energy together. The energy momentum relation for ideal gas is given by  $\varepsilon = \frac{p^2}{2m}$ , where  $m$  is the mass of each particle. Then for sufficient large volume the summation in relation (20) can be replaced by an integral over all phase space with the help of density of states in GUP framework (10). By substituting (10) in (20), we have

$$N_e^{cl} \simeq z \int_V \int_0^\infty \frac{d^3X d^3P}{[h(1 + \beta P^2)]^3} e^{-\frac{P^2}{2mT}}. \quad (23)$$

The integral over coordinates gives the three dimensional volume element  $V$  and we get

$$N_e^{cl} = \frac{Vz}{4h^3} \left( \frac{\pi}{\beta} \right)^{3/2} \left[ \frac{\sqrt{2\gamma}(1 + \gamma)}{\gamma^2} + \sqrt{\pi} e^{1/\gamma} \frac{(\gamma^2 - 2\gamma - 1)}{\gamma^2} \left( 1 - \operatorname{erf}\left(\frac{1}{\sqrt{\gamma}}\right) \right) \right], \quad (24)$$

where  $\gamma = \beta mT$  and  $\operatorname{erf}$  is the error function. Here  $N_e^{cl}$  denotes the number of excited particles for ideal gas in gravitational statistics. We can remove the gravity effects in the limit  $\beta \rightarrow 0$

$$\lim_{\beta \rightarrow 0} N_e^{cl} = \frac{V}{\lambda^3} z, \quad (25)$$

where  $\lambda = \frac{h}{\sqrt{2\pi mT}}$  is the mean thermal wavelength, in agreement with usual statistical physics. Relation (24) has a interesting consequence in the limit of  $T \rightarrow \infty$  as

$$\lim_{T \rightarrow \infty} N_e^{cl} = \frac{Vz}{4h^3} \frac{\pi^2}{\beta^{3/2}}. \quad (26)$$

The above relation shows that there is an upper bound for the number of excited particles in the limit of very high temperature which is a purely quantum gravity effect and does not have a classical analogy. More precise, chemical potential is the mean energy which one particle need to apply to statistical system, then if we increase the number of particles, the energy of the system becomes large and larger. In ordinary Maxwell-Boltzmann statistics, the particles access to the upper energy levels in the limit of high temperature and there is not any limitation for them. However in GUP theory, there is a highest energy level for systems. Then, after the energy of the system reaches to the highest level, if we increase the number of particles, these should be lie in ground state and we have a condensation for classical ideal gas. Such a condensation is different from ordinary ones in Maxwell-Boltzmann statistics which is occurs in the limit of low temperature. By having  $N_e^{cl}$  from relation (24), the pressure of the system can be obtained from relation (22) as

$$\mathcal{P}^{cl} = \frac{z \pi^{3/2}}{4mh^3} \frac{1}{\beta^{5/2}} \left[ \frac{\sqrt{2\gamma}(1 + \gamma)}{\gamma} + \sqrt{\pi} e^{1/\gamma} \frac{(\gamma^2 - 2\gamma - 1)}{\gamma} \left( 1 - \operatorname{erf}\left(\frac{1}{\sqrt{\gamma}}\right) \right) \right]. \quad (27)$$

Also, the energy of the system can be obtained by definition

$$\frac{U}{V} = T^2 \left[ \frac{\partial}{\partial T} \left( \frac{\mathcal{P}}{T} \right) \right]_z, \quad (28)$$

from which we get

$$U^{cl} = \frac{zV \pi^{3/2}}{8mh^3} \frac{1}{\beta^{5/2} \gamma^2} \left[ 2\sqrt{\pi} e^{1/\gamma} \sqrt{\gamma} (\gamma^2 + 4\gamma + 1) \operatorname{erfc}\left(\frac{1}{\sqrt{\gamma}}\right) - \gamma \left( (\sqrt{2} - 2) \gamma^2 + (3\sqrt{2} + 4) \gamma + 2 \right) \right], \quad (29)$$

where  $\operatorname{erfc}$  is the complementary error function. The pressure and energy of the classical ideal gas in purely classical limit (quantum effects are negligible) do not show the unusual behavior and have ordinary interpretation.

### 3.2 Extreme relativistic gas

As same as pervious section we consider  $N$  non-interacting particles which confined in volume  $V$ , but here the energy momentum relation for extreme relativistic gas is given by  $\varepsilon = P$ . For large volume, we can replace summation in (20) with the help of density of state (10) and we have

$$N_e^{cl} = \frac{Vz}{h^3} \frac{\sqrt{\pi}}{\beta^{3/2}} G_{0\ 0}^{3\ 1} \left( \frac{1}{4\beta T^2} \left| \begin{array}{c} \frac{-1}{2} \\ 0 \end{array} \right. \begin{array}{cc} \frac{1}{2} & \frac{3}{2} \end{array} \right), \quad (30)$$

where  $G$  is the Meijer function. Like the ideal gas, we find a condensation due to the GUP theory for extreme relativistic gas in the limit of high temperature

$$\lim_{T \rightarrow \infty} N_e^{cl} = \frac{Vz}{4h^3} \frac{\pi^2}{\beta^{3/2}}. \quad (31)$$

Thus, again there is an upper limit for the number of particles in the limit of high temperature  $T \rightarrow \infty$ . The maximum value of the number of particles for extreme relativistic gas is as same as the ideal gas (see relation (26)). The pressure can be obtained by substituting (30) in (22). For energy  $U^{cl}$  of the system we have

$$U^{cl} = \frac{\sqrt{\pi}}{2h^3 \beta^{5/2}} \frac{zV}{T} G_{0\ 0}^{3\ 1} \left( \frac{1}{4\beta T^2} \left| \begin{array}{c} \frac{-3}{2} \\ \frac{-1}{2} \end{array} \right. \begin{array}{cc} 0 & \frac{1}{2} \end{array} \right). \quad (32)$$

In the limit of high temperature  $T \rightarrow \infty$ , for the above relation we have

$$U^{cl} = \frac{Vz}{h^3} \frac{\pi}{\beta^2}. \quad (33)$$

Thus, there is an upper bound for energy of the extreme relativistic gases in GUP framework which is as order of Planck energy  $U^{cl} \sim E_{pl}$  (note  $\beta \sim \frac{1}{E_{pl}^2}$  in our unit).

## 4 Fermi-Dirac and Bose-Einstein statistics

In this limit the type of particles, which may be fermion or boson, plays an important role, according to which there are two types of statistics the so-called Fermi-Dirac and Bose-Einstein for fermions and bosons respectively. To consider the thermodynamics of fermions and bosons, we start again with relations (17) and (18). We replace summation in these relations by integral over all deformed phase space with the help of relation (10). As we mentioned, these integrals do not have a closed form solution, but fortunately they have an analytical solution in the limit of  $\beta \rightarrow 0$  (which is a good



approximation, because of the relation  $\beta \sim \frac{1}{E_{pl}^2}$ ) and we calculate them in terms of Fermi-Dirac and Bose-Einstein functions for general power law energy-momentum relation  $\varepsilon = \eta P^\alpha$  in Appendix B. The number of excited particles and pressure for fermions and bosons (see relations (B-5) and (B-8)) are given by

$$N_e = \frac{4\pi g}{\alpha h^3} (T/\eta)^{3/\alpha} \Gamma(3/\alpha) h_{3/\alpha}(z) \left[ 1 - 3\beta (T/\eta)^{2/\alpha} \frac{\Gamma(5/\alpha) h_{5/\alpha}(z)}{\Gamma(3/\alpha) h_{3/\alpha}(z)} + 6\beta^2 (T/\eta)^{4/\alpha} \frac{\Gamma(7/\alpha) h_{7/\alpha}(z)}{\Gamma(3/\alpha) h_{3/\alpha}(z)} + \dots \right], \quad (34)$$

and

$$\frac{\mathcal{P}}{T} = \frac{4\pi g}{\alpha h^3} (T/\eta)^{3/\alpha} \Gamma(3/\alpha) h_{\frac{3}{\alpha}+1}(z) \left[ 1 - 3\beta (T/\eta)^{2/\alpha} \frac{\Gamma(5/\alpha) h_{\frac{5}{\alpha}+1}(z)}{\Gamma(3/\alpha) h_{\frac{3}{\alpha}+1}(z)} + 6\beta^2 (T/\eta)^{4/\alpha} \frac{\Gamma(7/\alpha) h_{\frac{7}{\alpha}+1}(z)}{\Gamma(3/\alpha) h_{\frac{3}{\alpha}+1}(z)} + \dots \right], \quad (35)$$

where  $g$  is a weight factor showing the internal degree of freedom which for bosons and spin-1/2 particles is equal 1 and 2 respectively. Here,  $h_\nu(z)$  is given by relation (B-4) and reduces to Fermi-Dirac and Bose-Einstein functions according to the type of particles. Now, we can obtain energy of the systems by using of the definition (28) as

$$U = \frac{3VT}{\alpha} \frac{4\pi g}{\alpha h^3} (T/\eta)^{3/\alpha} \Gamma(3/\alpha) h_{\frac{3}{\alpha}+1}(z) \left[ 1 - 5\beta (T/\eta)^{2/\alpha} \frac{\Gamma(5/\alpha) h_{\frac{5}{\alpha}+1}(z)}{\Gamma(3/\alpha) h_{\frac{3}{\alpha}+1}(z)} + 14\beta^2 (T/\eta)^{4/\alpha} \frac{\Gamma(7/\alpha) h_{\frac{7}{\alpha}+1}(z)}{\Gamma(3/\alpha) h_{\frac{3}{\alpha}+1}(z)} + \dots \right]. \quad (36)$$

We can combine relations (35) and (36) to get equation of state for power law energy-momentum statistical systems

$$\frac{\mathcal{P}}{\rho} = \frac{\alpha}{3} \left[ 1 + 2\beta (T/\eta)^{2/\alpha} \frac{\Gamma(5/\alpha) h_{\frac{5}{\alpha}+1}(z)}{\Gamma(3/\alpha) h_{\frac{3}{\alpha}+1}(z)} + 2\beta^2 (T/\eta)^{4/\alpha} \left( 5 \left( \frac{\Gamma(5/\alpha) h_{\frac{5}{\alpha}+1}(z)}{\Gamma(3/\alpha) h_{\frac{3}{\alpha}+1}(z)} \right)^2 - 4 \frac{\Gamma(7/\alpha) h_{\frac{7}{\alpha}+1}(z)}{\Gamma(3/\alpha) h_{\frac{3}{\alpha}+1}(z)} \right) + \dots \right], \quad (37)$$

where  $\rho = U/V$  is the energy density. The l.h.s. of above equation denotes the equation of state (EoS) parameter  $\omega$  which is a function of temperature in GUP framework. The first term in the r.h.s. is an ordinary EoS parameter and the other terms exhibit the quantum gravity effects which, in a cosmological point of view, are very small in the late times of cosmic evolution. However, in the planck scale they become important and play a determinant role in early universe thermodynamics. Then, we rewrite the above equation in the form

$$\omega(\beta, T) = \omega_0 \left[ 1 + 2\beta (T/\eta)^{2/\alpha} \frac{\Gamma(5/\alpha) h_{\frac{5}{\alpha}+1}(z)}{\Gamma(3/\alpha) h_{\frac{3}{\alpha}+1}(z)} + 2\beta^2 (T/\eta)^{4/\alpha} \left( 5 \left( \frac{\Gamma(5/\alpha) h_{\frac{5}{\alpha}+1}(z)}{\Gamma(3/\alpha) h_{\frac{3}{\alpha}+1}(z)} \right)^2 - 4 \frac{\Gamma(7/\alpha) h_{\frac{7}{\alpha}+1}(z)}{\Gamma(3/\alpha) h_{\frac{3}{\alpha}+1}(z)} \right) + \dots \right]. \quad (38)$$

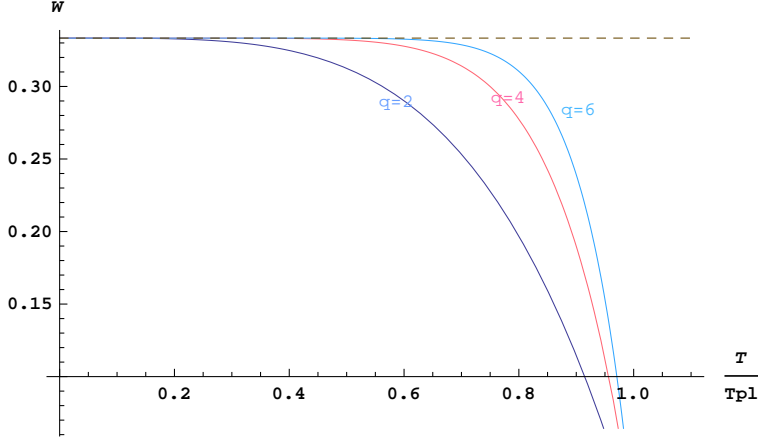


Figure 1: The effects of the minimal length on the EoS parameter  $\omega$ . As we see in the figure, the EoS parameter is a function of temperature in the GUP framework and get a negative value if temperature becomes more than critical temperature  $T_c = (\frac{1}{\sigma})^{\frac{1}{2q}} T_{Pl}$ .

Now, at the first, we consider above relation for non-relativistic particles. The energy-momentum relation for the non-relativistic particles is given by relation  $\varepsilon = \frac{p^2}{2m}$ , where  $m$  is the mass of the particles ( $\alpha = 2$ ,  $\eta = \frac{1}{2m}$ ). In this case we have

$$\omega(T) \simeq \frac{2}{3} \left[ 1 + 5m \frac{T}{T_{Pl}^2} \right]. \quad (39)$$

It is clear that even at the Planck temperature  $T = T_{Pl}$ , the second term in the r.h.s. of this relation has a small value because of the large value of the Planck temperature  $T_{Pl} \simeq 1.221 \times 10^{19} \text{ GeV}$ . But for relativistic particles the energy-momentum relation is given by  $\varepsilon = P$  ( $\alpha = 1 = \eta$ ) and relation (38) for the bosonic gas in the limit  $z \rightarrow 1$  becomes

$$\omega(T) = \frac{1}{3} \left[ 1 + 24 \frac{\zeta(6)}{\zeta(4)} \left( \frac{T}{T_{Pl}} \right)^2 + 1440 \left[ \frac{\zeta(6)}{\zeta(4)} \left( \frac{\zeta(6)}{\zeta(4)} - 2 \frac{\zeta(8)}{\zeta(6)} \right) \right] \left( \frac{T}{T_{Pl}} \right)^4 + \dots \right], \quad (40)$$

where  $\zeta(\nu)$  is the Riemann zeta function. Unlike the case of non-relativistic particles, the above relation does not converge in the limit of high temperature and all terms in the r.h.s. become important. Since the zeta function decreases with its argument, the coefficients of all the terms in r.h.s. except the second one are negative. In this sense we offer the simplest temperature power law for the relation (40) as

$$\omega(T) \equiv \frac{1}{3} \left[ 1 - \sigma \left( \frac{T}{T_{Pl}} \right)^{2q} \right], \quad (41)$$

where  $\sigma$  and  $q \geq 2$  are some constants which may be fixed by cosmological observation. An interesting feature of relation (41) is when the temperature increases and becomes larger than critical temperature  $T_c = (\frac{1}{\sigma})^{\frac{1}{2q}} T_{Pl}$ , for which we have a quintessence scenario for boson gaseous system. If we set  $T_c = T_{Pl}$  then relation (41) describes an inflationary scenario. In figure 1 we have plotted the behavior of  $\omega$  for three values of  $q$  in which the dashed line shows the ordinary case and three curves show the effects of minimal length on cosmological parameter  $\omega$  as a function of temperature. It is clear that in the limit of low temperature all the curves tend to ordinary case which shows that the quantum gravity effects only become significant in high temperature i.e. in the Planck scales.

## 4.1 Bose-Einstein condensation

After set up the Bose-Einstein statistics, it is natural to consider famous phenomenon Bose-Einstein condensation. The equation (34) for bosons gives

$$N_e = \frac{4\pi}{\alpha h^3} (T/\eta)^{3/\alpha} \Gamma(3/\alpha) g_{3/\alpha}(z) \left[ 1 - 3\beta (T/\eta)^{2/\alpha} \frac{\Gamma(5/\alpha) g_{5/\alpha}(z)}{\Gamma(3/\alpha) g_{3/\alpha}(z)} + 6\beta^2 (T/\eta)^{4/\alpha} \frac{\Gamma(7/\alpha) g_{7/\alpha}(z)}{\Gamma(3/\alpha) g_{3/\alpha}(z)} + \dots \right]. \quad (42)$$

We know from the ordinary case that the fugacity of the bosonic systems has a finite rang  $0 < z < 1$  and Bose-Einstein condensation occurs in the limit of  $z \rightarrow 1$ . Then, in this limit the above relation becomes

$$N_e^{max} = N_{0e}^{max} \left[ 1 - 3\beta (T/\eta)^{2/\alpha} \frac{\Gamma(5/\alpha)\zeta(5/\alpha)}{\Gamma(3/\alpha)\zeta(3/\alpha)} + 6\beta^2 (T/\eta)^{4/\alpha} \frac{\Gamma(7/\alpha)\zeta(7/\alpha)}{\Gamma(3/\alpha)\zeta(3/\alpha)} + \dots \right], \quad (43)$$

where  $N_{0e}^{max} = \frac{4\pi}{\alpha h^3} (T/\eta)^{3/\alpha} \Gamma(3/\alpha) \zeta(3/\alpha)$  is the maximum number of excited particles in ordinary Bose-Einstein statistics. It is clear that the remaining terms in the r.h.s. are the quantum gravity effects and are small in low temperature. Therefore, for the non-relativistic particles with energy-momentum relation  $\varepsilon = \frac{P^2}{2m}$ , we get

$$N_e^{max} \simeq \frac{(2\pi m T)^{\frac{3}{2}}}{h^3} \zeta(3/2) \left[ 1 - 6m \frac{T}{T_{Pl}^2} \right]. \quad (44)$$

Thus, the number of excited particles is smaller than the ordinary case and Bose-Einstein condensation occurs sooner in comparison with the ideal gas. However, the correction term due to the GUP effects is very small even in Planck Temperature. For relativistic particles with energy-momentum relation  $\varepsilon = P$ , relation (43) becomes

$$N_e^{max} \simeq \frac{8\pi}{h^3} T^3 \zeta(3) \left[ 1 - 31 \left( \frac{T}{T_{Pl}} \right)^2 + 1812 \left( \frac{T}{T_{Pl}} \right)^4 + \dots \right]. \quad (45)$$

The above relation does not converge in high temperature and all the correction terms in the r.h.s. are important.

## 4.2 Degenerate gas

One of the interesting cases in Fermi-Dirac statistics is a completely degenerate gas which corresponds to the limit  $T \rightarrow 0$ . In this limit the mean occupation number  $\langle n_\varepsilon \rangle$  in relation (16) takes a simple form as

$$\langle n_\varepsilon \rangle = \begin{cases} 1 & \varepsilon < \mu_0 \\ 0 & \varepsilon > \mu_0, \end{cases}$$

where  $\mu_0$  denotes to the chemical potential in the limit  $T \rightarrow 0$ . Then, mean occupation number becomes independent of the temperature and acts as a step function. Now, the relations (17) and (18) become very simple and have an exact solution also in GUP framework. Then, relation (17) takes the form

$$N = \sum_{\varepsilon} \simeq \int a(\varepsilon) d\varepsilon. \quad (46)$$

Here, there is a limited value for energy which is called Fermi energy  $\varepsilon_F$ , in terms of which one can also define the Fermi momentum  $P_F$ . Upon using of relation (10), we have

$$N = \frac{4\pi g V}{h^3} \int_0^{P_F} \frac{P^2 dP}{(1 + \beta P^2)^3} = \frac{\pi V g}{2h^3} \left[ \frac{\arctan(\sqrt{\beta} P_F)}{\beta^{\frac{3}{2}}} + \frac{P_F(\beta P_F^2 - 1)}{\beta(1 + \beta P_F^2)^2} \right], \quad (47)$$

which denotes to the number of particles in completely degenerate gaseous system with any energy-momentum relation in GUP framework. Up to the first order of  $\beta$  from the above relation one obtains

$$\frac{9}{5}\beta(P_F)^5 - (P_F)^3 + P_{0_F}^3 = 0, \quad (48)$$

where  $P_{0_F} = (\frac{3N}{4\pi V g})^{\frac{1}{3}} h$  is the Fermi momentum in ordinary case. We get  $P_F = P_{0_F}$  in the limit  $\beta = 0$ , so we expand the above relation in the small but nonzero limit of GUP parameter  $\beta \rightarrow 0$  to get

$$P_F \simeq P_{0_F} \left[ 1 + \frac{3}{5}\beta P_{0_F}^2 - \frac{24}{175}\beta^2 P_{0_F}^4 + \dots \right]. \quad (49)$$

The first term in r.h.s. of the above relation denotes the ordinary Fermi momentum and the second term is the most important effect of GUP on the Fermi momentum. Then, independent of that the gaseous system being relativistic or not, minimal length effects increase the value of the Fermi momentum and consequently increase the Fermi energy.

## 5 Conclusions

There are some evidences which support the idea of the existence of minimal length from quantum gravity theories. GUP theory imposes minimal length in quantum mechanics by modifying uncertainty relations, then the deformed Heisenberg algebra governs the phase space instead of the usual Heisenberg algebra. The customary procedure for exerting GUP conditions to a physical systems is finding the generalized coordinates and momenta which satisfy the deformed Heisenberg algebra. For the simplest choice, the Schrödinger equation becomes non-linear or it becomes a higher order differential equation and it is not easy to solve it. In this paper we have extended statistical mechanics in GUP framework without fixing coordinates and momenta. We investigated the possibility of existence a general transformation in phase space which changes the usual Heisenberg algebra to a deformed one. Such a transformation changes the density of states in phase space. We obtained a deformed density of states for GUP conditions by the use of the Jacobian of the transformation. In this picture, effects of minimal length only changes the structure of phase space, so we conclude minimal length influences the situations of microstates in quantum scales and decreases the number of accessible microstates of the system. The advantage of this procedure is that we do not need to solve the modified Schrödinger equation, instead we can work with the classical Hamiltonian together deformed density of state. In this method, we do not use the wave functions, so the exact solution of modified Schrödinger equations reserve their importance. Also we showed the non-commutativity of coordinates due to a GUP condition does not contribute in the density of states till the momenta commute. We develop the ensemble theory by this approach and we study the Maxwell-Boltzmann, Fermi-Dirac

and Bose-Einstein statistics. We found an interesting phenomenon in Maxwell-Boltzmann statistics: there is an upper bound for the number of excited particles for gaseous system (ideal gas and extreme relativistic gas) in the limit of high temperatures, which means that we have a condensation due to a GUP theory. We also found an interesting temperature dependent equation of state parameter  $\omega$  for the bosonic gas based on which our analysis showed these effects are negligible for non-relativistic particles even in Planck scales. On the other hand, these effects become very important for the relativistic bosonic gaseous systems and lead to an inflationary scenario in the limit of high temperatures. We showed there is a critical temperature which can be determined in the limit of the acceleration phase, but there is two free parameters that can be fixed with the cosmological observations. We studied two interesting full quantum cases, Bose-Einstein condensation and the completely degenerate gas in the GUP framework. Our analysis shows the number of excited particles becomes smaller and the condensation occurs sooner. For the degenerate gas, we found a modified closed form for the number of particles in GUP framework and we obtained a modification to a Fermi energy and momentum. Finally we concluded all the corrections due to a minimal length are small in ordinary temperatures and will become important on the high temperature limit.

## A Jacobian in terms of Poisson brackets

In this Appendix, we would like to show that it is possible to define general transformation in phase space which transforms coordinates  $x_i$  and momenta  $p_i$  to generalized coordinates  $X_i$  and momenta  $P_i$  as new phase space variable satisfy all the relations (5), (6) and (7). It is well-known from classical physics that the coordinates  $x_i$  and momenta  $p_i$  satisfy the Poisson algebra  $\{x_i, p_i\} = \delta_{ij}$ . However, Darboux theorem implies that it is always possible to find variables  $X_i$  and  $P_i$  as a function of  $x_i$  and  $p_i$  which satisfy relation

$$\{X_i, P_j\} = f(X_i, P_j). \quad (\text{A-1})$$

Therefore, the relation (6) is a particular case of the Darboux theorem and in GUP framework  $f(X, P) = f(P) = i\hbar(1 + \beta P^2)$ . On the other hand, relation (7) does not exert any constraint to our study. Thus, we can deduce that the relations (6) and (7) are valid simultaneously in general. But we want to have a transformation which satisfy also relation (5). We suppose general forms of non-commutativity as

$$\{X_i, X_j\} = g(X_i, P_j), \quad (\text{A-2})$$

where  $g$  is the arbitrary function of variables  $X_i$  and  $P_j$ . Again, it is possible to find variables  $X_i$  and  $P_j$  as a function of  $x_i$  and  $p_i$  which satisfy relation (A-2), but we should have variables  $X_i$  and  $P_j$  as a function of  $x_i$  and  $p_j$  which satisfy relations (5), (6) and (7) simultaneously. In general, we cannot prove that it is always possible to define general transformation which changes ordinary variables  $x_i$  and  $p_j$  (which satisfy Poisson algebra) to GUP variables  $X_i$  and  $P_j$  (which satisfy deformed Poisson algebra). However, at least there is one choice  $X_i$  and  $P_i$  as

$$\begin{cases} X_i = x_i (1 + \beta p^2), \\ P_i = p_i, \end{cases}$$

which satisfy all the relations (5), (6) and (7) simultaneously. This implies to define transformation between usual coordinates  $(x_i, p_i)$  and GUP variables  $(X_i, P_i)$ . The Jacobian of transformation can be expanded in terms of Poisson brackets in  $2\mathcal{N}$ -dimensional phase space as [5]

$$J = \frac{\partial(X_i, P_i)}{\partial(x_i, p_i)} = \frac{1}{2^{\mathcal{N}} \mathcal{N}!} \sum_{i_1 \dots i_{2\mathcal{N}}=1}^{2\mathcal{N}} \varepsilon_{i_1 \dots i_{2\mathcal{N}}} \{J_{i_1}, J_{i_2}\} \dots \{J_{i_{2\mathcal{N}-1}}, J_{i_{2\mathcal{N}}}\}, \quad (\text{A-3})$$

where  $\varepsilon$  denotes the Levi-Civita symbol and  $J_i$  represents the phase space variables as for odd  $i$  is a coordinate  $X_i$  and for even  $i$  is a conjugate momentum  $P_i$ . In the above relation the Poisson brackets which denote only coordinates commutator (5), always multiply by Poisson brackets that purely denote the momentum commutator (7). Thus the coordinates Poisson brackets does not contribute because the momenta commute. Then only the Poisson brackets include both coordinates and momenta (6) contribute in relation (A-3) and Jacobian simplify to (9).

## B particle density and pressure in GUP framework

The number of excited particles in GUP framework can be obtained from relation (19), by substituting mean occupation number  $\langle n_\varepsilon \rangle$  from (16), we get

$$N_e = \int \frac{a(\varepsilon)d\varepsilon}{z^{-1}e^{\varepsilon/T} + \epsilon}, \quad (\text{B-1})$$

where  $a(\varepsilon)d\varepsilon$  is the density of states in GUP framework determined by (10). Evaluation of this integral depends on the energy-momentum relation which unfortunately, there is no closed form solution for familiar power law energy-momentum. So we expand  $a(\varepsilon)d\varepsilon$  for small  $\beta$ . Up to the second order of  $\beta$  in three dimension (10) we obtain

$$a(\varepsilon)d\varepsilon \simeq \frac{d^3X d^3P}{h^3} \left( 1 - 3\beta P^2 + 6\beta^2 P^4 + \dots \right). \quad (\text{B-2})$$

We consider general power law energy-momentum relation  $\varepsilon = \eta P^\alpha$  which reduces to the ultra-relativistic case for  $\eta = 1 = \alpha$  and get the non-relativistic energy-momentum relation for  $\eta = 1/2m$  and  $\alpha = 2$ . Then by substituting (B-2) in (B-1) results

$$\begin{aligned} \frac{N}{V} = n = \frac{4\pi}{\alpha h^3} (T/\eta)^{3/\alpha} & \left[ \int_0^\infty \frac{y^{\frac{3}{\alpha}-1} dy}{z^{-1}e^y + \epsilon} - 3\beta (T/\eta)^{2/\alpha} \int_0^\infty \frac{y^{\frac{5}{\alpha}-1} dy}{z^{-1}e^y + \epsilon} \right. \\ & \left. + 6\beta^2 (T/\eta)^{4/\alpha} \int_0^\infty \frac{y^{\frac{7}{\alpha}-1} dy}{z^{-1}e^y + \epsilon} + \dots \right], \end{aligned} \quad (\text{B-3})$$

where  $n$  is the particle density and  $y = \varepsilon/T$ . Integrals in the above relation are nothing but Fermi-Dirac and Bose-Einstein functions for  $\epsilon = 1$  and  $\epsilon = -1$  respectively. We define function  $h_\nu(z)$  as

$$h_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{y^{\nu-1} dy}{z^{-1}e^y + \epsilon} = \begin{cases} f_\nu(z) & \epsilon = +1 \\ g_\nu(z) & \epsilon = -1 \end{cases} \quad (\text{B-4})$$

where  $\Gamma(\nu)$  is a factorial function. Now particle density in terms of Fermi-Dirac and Bose-Einstein functions becomes

$$\begin{aligned} n = \frac{4\pi}{\alpha h^3} (T/\eta)^{3/\alpha} \Gamma(3/\alpha) h_{3/\alpha}(z) & \left[ 1 - 3\beta (T/\eta)^{2/\alpha} \frac{\Gamma(5/\alpha) h_{5/\alpha}(z)}{\Gamma(3/\alpha) h_{3/\alpha}(z)} \right. \\ & \left. + 6\beta^2 (T/\eta)^{4/\alpha} \frac{\Gamma(7/\alpha) h_{7/\alpha}(z)}{\Gamma(3/\alpha) h_{3/\alpha}(z)} + \dots \right]. \end{aligned} \quad (\text{B-5})$$

On the other hand, pressure of the system will be determined by relation (18). Again, for large volume we can replace summation by an integral. By using of relation (B-2) for general energy-momentum

relation we get

$$\begin{aligned} \frac{\mathcal{P}}{T} = \frac{4\pi}{\epsilon \alpha h^3 \eta^{3/\alpha}} & \left[ \int_0^\infty \ln(1 + \epsilon z e^{-\epsilon/T}) \epsilon^{\frac{3}{\alpha}-1} d\epsilon - 3\beta \eta^{-2/\alpha} \int_0^\infty \ln(1 + \epsilon z e^{-\epsilon/T}) \epsilon^{\frac{5}{\alpha}-1} d\epsilon \right. \\ & \left. + 6\beta^2 \eta^{-4/\alpha} \int_0^\infty \ln(1 + \epsilon z e^{-\epsilon/T}) \epsilon^{\frac{7}{\alpha}-1} d\epsilon + \dots \right]. \end{aligned} \quad (\text{B-6})$$

Integrating by part and set  $y = \epsilon/T$  yields

$$\begin{aligned} \frac{\mathcal{P}}{T} = \frac{4\pi}{3h^3} (T/\eta)^{3/\alpha} & \left[ \int_0^\infty \frac{y^{3/\alpha} dy}{z^{-1}e^y + \epsilon} - \frac{9}{5}\beta (T/\eta)^{2/\alpha} \int_0^\infty \frac{y^{5/\alpha} dy}{z^{-1}e^y + \epsilon} \right. \\ & \left. + \frac{18}{7}\beta^2 (T/\eta)^{4/\alpha} \int_0^\infty \frac{y^{7/\alpha} dy}{z^{-1}e^y + \epsilon} + \dots \right]. \end{aligned} \quad (\text{B-7})$$

The above integrals are again Fermi-Dirac and Bose-Einstein functions and we can express them in terms of  $h_\nu(z)$  as

$$\begin{aligned} \frac{\mathcal{P}}{T} = \frac{4\pi}{\alpha h^3} (T/\eta)^{3/\alpha} \Gamma(3/\alpha) h_{\frac{3}{\alpha}+1}(z) & \left[ 1 - 3\beta (T/\eta)^{2/\alpha} \frac{\Gamma(5/\alpha) h_{\frac{5}{\alpha}+1}(z)}{\Gamma(3/\alpha) h_{\frac{3}{\alpha}+1}(z)} \right. \\ & \left. + 6\beta^2 (T/\eta)^{4/\alpha} \frac{\Gamma(7/\alpha) h_{\frac{7}{\alpha}+1}(z)}{\Gamma(3/\alpha) h_{\frac{3}{\alpha}+1}(z)} + \dots \right]. \end{aligned} \quad (\text{B-8})$$

Relations (B-5) and (B-8) are essential for our study in thermodynamics of bosons and fermions in GUP framework.

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